ECON 408: HW 1

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1 By hand computation

$$Y = \begin{bmatrix} 8 \\ 12 \\ 12 \\ 16 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

For this assignment I need to compute the following:

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} (1+1+1+1) & (1+2+3+4) \\ (1+2+3+4) & (1+4+9+16) \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{|X'X|} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} = \frac{1}{120-100} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 12 \\ 16 \end{bmatrix} = \begin{bmatrix} (8+12+12+16) \\ (8+24+36+64) \end{bmatrix} = \begin{bmatrix} 48 \\ 132 \end{bmatrix}$$

$$\hat{\beta} = (X'X)^{-1}X'Y = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 48 \\ 132 \end{bmatrix} = \begin{bmatrix} 72 + (-66) \\ -24 + 26.4 \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{12}{5} \end{bmatrix}$$

$$e = Y - X\hat{\beta} = \begin{bmatrix} 8 \\ 12 \\ 12 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ \frac{12}{5} \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 12 \\ 16 \end{bmatrix} - \begin{bmatrix} 6 + \frac{12}{5} \\ 6 + \frac{24}{5} \\ 6 + \frac{36}{5} \\ 6 + \frac{48}{5} \end{bmatrix} = \begin{bmatrix} 8-6-2.4 \\ 12-6-4.8 \\ 12-6-7.2 \\ 16-6-9.6 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 6/5 \\ -6/5 \\ 2/5 \end{bmatrix}$$

$$\hat{\sigma}^2 = e'e/(df) = e'e/(4-2) = \frac{1}{2}e'e = \begin{bmatrix} -\frac{2}{5} & \frac{6}{5} & -\frac{6}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} -2/5 \\ 6/5 \\ -6/5 \\ 2/5 \end{bmatrix} = \frac{1}{2}\left(\frac{4}{25} + \frac{36}{25} + \frac{36}{25} + \frac{4}{25}\right) = \frac{8}{5} = \hat{\sigma}^2$$

$$Var(\hat{\beta}) = \hat{\sigma}^2(X'X)^{-1} = \frac{8}{5} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{12}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{8}{25} \end{bmatrix} = Var(\hat{\beta})$$

The standard errors are given by the square root of the diagonal of the matrix above. So:

$$\hat{\sigma} = \begin{bmatrix} \sqrt{\frac{12}{5}} \\ \sqrt{\frac{8}{25}} \end{bmatrix}$$

The T-Ratios are calculated using $\hat{\beta}$ and the standard error

$$\text{T-Ratio} = \frac{\hat{\beta}_{ij}}{\hat{\sigma}_{ij}} = \begin{bmatrix} 6 \\ \frac{12}{5} \end{bmatrix}_{ij} \div \begin{bmatrix} \sqrt{\frac{12}{5}} \\ \sqrt{\frac{8}{25}} \end{bmatrix}_{ij} = \begin{bmatrix} \sqrt{15} \\ \frac{15}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 3.87 \\ 10.61 \end{bmatrix}$$

2 Real data

I did all of these in both Stata and R. The Stata do-file code is attached. I did it in R also because I wanted to be more familiar with matrix algebra in R, and the fact that I'm writing this assignment using R Markdown, making it easier to work with for outputting clean looking matrices. The code does the same thing for both though, so I just included both.

Stata Do-file:

```
clear all
cd "/Users/liz/Documents/Projects/ECON 408/HW1"
*Import Data
import excel "hw1.xls", firstrow
* Create vector of ones for constant, create matricies
gen allone = 1
mkmat crcostpc, matrix(Y)
mkmat allone p1524 prisonpc, matrix(X)
*Calculate X'X, X'y, (X'X)^-1, and Beta Hat
mat XpX = X' * X
mat XpY = X' * Y
mat Bhat = inv(XpX) * XpY
mat Yhat = X * Bhat
*Calculate residuals, variance, and the variance covariance matrix
mat resid = Y - Yhat
scalar df = rowsof(X) - colsof(X)
mat variance = (resid' * resid)/(df)
mat varbhat = variance * inv(XpX)
*Extract diagonal entries of variance/covariance matrix
mat diag = vecdiag(varbhat)'
*Take square roots of the vector of the diagonal
matmap diag SE, map(sqrt(@))
*Use MATA to do element-wise division to calculate the T-ratios for each Beta hat
mata: st_matrix("Tratio", st_matrix("Bhat") :/ st_matrix("SE"))
```

Input the data into matrices

```
excel <- read_excel("hw1.xls")
X <- matrix(c(rep(1,74), excel$p1524, excel$prisonpc), ncol=3)
Y <- matrix(c(excel$crcostpc), ncol=1)
```

Calculate $X'X, X'y, (X'X)^{-1}$, and $\hat{\beta}$

```
XpX <- t(X) %*% X
XpY <- t(X) %*% Y
Bhat <- solve(XpX) %*% XpY
```

$$X'X = \begin{bmatrix} 74 & 1193.454 & 15294.474 \\ 1193.454 & 19571.516 & 234121.184 \\ 15294.474 & 234121.184 & 4687470.195 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 20244.646 \\ 330432.551 \\ 4421245.555 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} -233.489 \\ 26.644 \\ 0.374 \end{bmatrix}$$

Calculate $\hat{Y}$, the residuals, the degrees of freedom, and $\hat{\sigma}^2$. Residuals not displayed as they are too many to fit on the paper.

```
Yhat <- X %*% Bhat
resid = Y - Yhat
ESS <- t(resid) %*% resid
df = nrow(X) - ncol(X)
variance <- (t(resid) %*% resid) / df
```

$$\hat{\sigma}^2 = 4440.5313786$$

$$ESS = e'e = \sum_{i-1}^{N} e_i^2 = 3.1527773 \times 10^5$$

Calculate the variance/covariance matrix, extract the diagonal and take the square root to get the standard error.

```
varbhat <- drop(variance) * solve(XpX)
diag <- diag(varbhat)
SE <- sqrt(diag)
```

$$Var(\hat{\beta}) = \begin{bmatrix} 6578.561 & -358.698 & -3.549 \\ -358.698 & 20.122 & 0.165 \\ -3.549 & 0.165 & 0.004 \end{bmatrix}$$

$$\hat{\sigma} = \begin{bmatrix} 81.108 \\ 4.486 \\ 0.065 \end{bmatrix}$$

Calculate the T-Ratios for each of the elements of $\hat{\beta}$

```
T_ratio <- Bhat / SE
```

$$\text{T-Ratios} = \begin{bmatrix} -2.879 \\ 5.94 \\ 5.729 \end{bmatrix}$$

With this data, we can confidently reject the null hypothesis, assuming the MLR assumptions are justified. Both p1524 and prisonpc have coefficents are statistically significantly greater than zero. This means we can reject the null hypothesis. The standard errors were very small compared to the coefficients, and even on a lower bound of the confidence interval has it well above zero.